An open problem on reordered computable numbers

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Let X be a set. A *representation* of X is a partial function $\rho\colon \subseteq \mathbb{N}^{\mathbb{N}} \to X$ which is a surjective. On the set of positive real numbers we can define the following representation. A function $f\colon \mathbb{N} \to \mathbb{N}$ is called a *name* for a real number $x > 0$ if the dyadic series $\sum_{k=0}^{\infty} 2^{-f(k)}$ converges to x. In particular, f tends to infinity, and every real number has infinitely many names. It is easy to see that a real number is left-computable if and only if it has a computable name. Therefore, the left-computable numbers are the computable points in this represented space.

Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function which tends to infinity. Then we can define the function $u_f\colon \mathbb{N} \to \mathbb{N}$ by

$$u_f(n) := |\{k \in \mathbb{N} \mid f(k) = n\}| \tag{1}$$

for all $n \in \mathbb{N}$. Now suppose that the series $\sum_{k=0}^{\infty} 2^{-f(k)}$ converges. Then the reordered series $\sum_{k=0}^{\infty} u_f(k) \cdot 2^{-k}$ converges to the same limit. We know from classical analysis that in this case we have $\limsup_{n\to\infty} \sqrt[n]{u_f(n)} \in [1, 2]$.

For every left-computable number x, we can define

$$\varsigma_x := \inf\{\limsup_{n\to\infty} \sqrt[n]{u_f(n)} \mid f \text{ is a computable name for } x\} \tag{2}$$

the *characteristic dyadic value* of x. The intuition behind this value is that the higher it is, the less efficient are the computable dyadic approximations for this number. It can be shown [1] that we have $\varsigma_x = 1$ for every computable number and $\varsigma_x = 2$ for every Martin-Löf random number.

Furthermore, it can be shown [1] that for every computably approximable number $r \in [1, 2]$ there exist a left-computable number x with $\varsigma_x = r$ such that there exists a computable name $f\colon \mathbb{N} \to \mathbb{N}$ for x with $\limsup_{n\to\infty} \sqrt[n]{u_f(n)} = r$, hence for this number x the infimum in Definition 2 is a actually a minimum.

My open question is the following: Is the infimum in Definition 2 always a minimum or do a real number $s \in [1, 2]$ and a left-computable number y with $\varsigma_y = s$ and $\limsup_{n\to\infty} \sqrt[n]{u_f(n)} > s$ for every computable function $f\colon \mathbb{N} \to \mathbb{N}$ which is a name for x?

References

[1] P. Janicki. Reordered computable numbers, 2024.